HOMEWORK 7 SOLUTIONS

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Problem 10.1

First suppose there exists a bijection $f : X \to Y$. Since X is a nonempty finite set of cardinality n, there exists a bijection $g : \mathbb{N}_n \to X$. Then $g^{-1} \circ f^{-1} : Y \to \mathbb{N}_n$ is a bijection, since the composition of bijections is a bijection. Now suppose $|Y| = n$. Then there exists a bijection $f : \mathbb{N}_n \to Y$, so $f \circ g^{-1} : X \to Y$ is a bijection.

Problem 11.1

We'll proceed by induction. For the base case, let $n = 1$, and suppose $f : \mathbb{N}_1 \to X$ is a surjection. Then $X = im(f) = \{f(1)\}$ which has a single element, so $|X| = n$. Now suppose that if there exists a surjection $\mathbb{N}_k \to X$ for some set X and integer $k \geq 1$, then $|X| \leq k$. Let $g : \mathbb{N}_{k+1} \to Y$ be a surjection. Consider the function $h : \mathbb{N}_k \to im(h)$ defined by $h(i) = g(i)$. Then h is certainly a surjection since any function is a surjection onto its image. By the induction hypothesis, we have $|im(h)| \leq k$. Furthermore,

$$|X| = |\{g(x)|x \in \mathbb{N}_{k+1}\}| = |\{g(x)|x \in \mathbb{N}_k\} \cup \{g(k+1)\}| \leq |\{g(x)|x \in \mathbb{N}_k\}| + 1 = |im(h)| + 1 \leq k + 1.$$

Thus, $|X| \leq k + 1$ as desired.

Problem 11.5

(i) First suppose $X = Y$. Since cardinality is well-defined, we must have $|X| = |Y|$.

Now suppose $|X| = |Y|$. Since $X \subseteq Y$, it is enough to show that $Y - X$ is empty. To see this, observe that $Y = X \cup (Y - X)$. Since these are disjoint sets, we have

$$|Y| = |X| + |Y - X| = |Y| + |Y - X|.$$

Thus we must have $|Y - X| = 0$, so $Y - X = \varnothing$.

(ii) Suppose $X \subset Y$. Define a function $f : X \to Y$ by $f(a) = a$ for all $a \in X$. Then f is an injection, so by the pigeonhole principle, $|X| \leq |Y|$. We can't have $|X| = |Y|$ since part (i) would have implied that $X = Y$. Thus, we must have $|X| < |Y|$.

Now suppose $|X| < |Y|$. Since $X \subseteq Y$ we have either $X = Y$ or $X \subset Y$. If $X = Y$ then part (i) would imply that $|X| = |Y|$. Since this is not the case, we must have $X \subset Y$.

Problem 1

Let R be the set of students who like reasoning, A be the set of students who like algebra, and C be the set of students who like calculus. There are 182 math students total, and $|R \cup A \cup C|$ students who like at least one of the subjects, so we need to compute $182 - |R \cup A \cup C|$. By the principle of inclusion-exclusion,

$$\begin{aligned}|R \cup A \cup C| &= |R| + |A| + |C| - |R \cap A| - |R \cap C| - |A \cap C| + |R \cap A \cap C| \\ &= 129 + 129 + 129 - 85 - 89 - 86 + 54 \\ &= 181.\end{aligned}$$

Thus we have $182 - |R \cup A \cup C| = 182 - 181 = 1$, so there is exactly one student who doesn't like any of the core modules.

Problem 4

We'll proceed by induction on n. When $n = 1$ there is a single nonempty subset of $\mathbb{N}_1$, namely $\{1\}$. Thus we have

$$\sum_{\varnothing \neq I \subseteq \mathbb{N}_n} (-1)^{|I|-1} |A_I| = (-1)^0 |A_{\{1\}}| = |A_1|$$

as desired. Now suppose

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{\varnothing \neq I \subseteq \mathbb{N}_n} (-1)^{|I|-1} |A_I|$$

for some integer $k \geq 1$. Recall that for any sets A and B, we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Then we have

$$\begin{aligned}\left| \bigcup_{i=1}^{k+1} A_i \right| &= \left| \bigcup_{i=1}^{k} A_i \right| + |A_{k+1}| - \left| A_{k+1} \cap \bigcup_{i=1}^{k} A_i \right| \\ &= \left| \bigcup_{i=1}^{k} A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^{k} (A_{k+1} \cap A_i) \right| \\ &= \sum_{\varnothing \neq I \subseteq \mathbb{N}_k} (-1)^{|I|-1} |A_I| + |A_{k+1}| - \sum_{\varnothing \neq I \subseteq \mathbb{N}_k} (-1)^{|I|-1} |A_I \cap A_{k+1}|\end{aligned}$$

For the second equality, we need to justify that

$$A_{k+1} \cap \bigcup_{i=1}^{k} A_i = \bigcup_{i=1}^{k} (A_{k+1} \cap A_i).$$

To see this, observe that x is an element of the left hand side if and only if x is in A_{k+1} and x is in $\bigcup_{i=1}^{k} A_i$, which means that x is in A_{k+1} and A_m for some $1 \leq m \leq k$. In other words, $x \in A_{k+1} \cap A_m$, which is equivalent to saying x is an element of the right hand side.

The last equality was justified by applying our induction hypothesis twice. In the second of those, the summands will look like $(-1)^{|I|-1} |\cap_{i \in I} (A_i \cap A_{k+1})| = (-1)^{|I|-1} |A_I \cap A_{k+1}|$, as claimed.

Let's take a look at what's going on here. We can think of the sets I being summed over in the first sum as nonempty subsets of $\mathbb{N}_{k+1}$ which don't contain $k+1$. In the second sum, note that $A_I \cap A_{k+1} = A_{I \cup \{k+1\}}$, so we can write that term instead as

$$\sum_{\varnothing \neq I \subseteq \mathbb{N}_k} (-1)^{|I|-1} |A_{I \cup \{k+1\}}|.$$

In other words, we interested in the cardinalities of the sets A_S where S is a subset of $\mathbb{N}_{k+1}$ which contains $k+1$. The only set the sum misses is $\{k+1\}$, which is accounted for by the term $|A_{k+1}|$. Since every nonempty subset of $\mathbb{N}_{k+1}$ either does or does not contain $k+1$, we can combine the terms to obtain

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{\varnothing \neq I \subseteq \mathbb{N}_{k+1}} (-1)^{|I|-1} |A_I|.$$

Problem 14

Let $f : A \to \{1, 3, 5, \ldots, 2n-1\}$ be the function such that $f(a)$ is the greatest odd integer which divides a. Since $|A| = n+1$ and $|\{1, 3, 5, \ldots, 2n-1\}| = n$, the pigeonhole principle implies that f is not an injection, so there exist a and b in A such that $f(a) = f(b)$. Given an integer x, we can write $x = 2^k f(x)$ for some integer k. This comes from the fact that we can write down the prime factorization of any integer. For example, $60 = 2 \cdot 2 \cdot 3 \cdot 5$. The largest prime factor of 60 is 15, and we have $60 = 2^2 \cdot 15 = 2^2 f(60)$. In other words, $f(x)$ is just the product of all the odd prime factors of f. Remember, the product of odd number is odd, so the product of the odd prime factors is also odd. Since 2 is the only even prime number, the product of the even factors will always look like 2^k for some integer k.

Back to our original argument, we know that $f(a) = f(b)$. Write $a = 2^k f(a)$ and $b = 2^j f(b)$. Without loss of generality, suppose $k > j$. (We know they're not equal since $a \neq b$.) Then we have $a = 2^{k-j} \cdot 2^j f(a) = 2^{k-j} \cdot 2^j f(b) = 2^{k-j} b$, so $b|a$.

Additional Problem

Let $A_i = \{2i-1, 2i\}$ and $B = \{A_1, A_2, \ldots, A_n\}$. Define a function $f : A \to B$ such that $f(x) = A_i$ if $x \in A_i$. For instance, $f(1) = A_1$ and $f(4) = A_2$. Since $|A| = n+1 > n = |B|$, the pigeonhole principle tells us that f is not injective, so there must exist x and y such that $f(x) = f(y)$. In other words, x and y are both in $\{2i-1, 2i\}$ for some integer i. Since $x \neq y$, we know x and y must be consecutive integers.